Flavor Condensate and Vacuum (In-)Stability in QED$_{2+1}$

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Abstract

In this letter we study (2+1)–dimensional QED. The first part contains the computation of the flavor symmetry–breaking condensate and its relation to the trace of the energy–momentum tensor, while the second part is concerned with the computation of the effective action allowing for non-constant static external magnetic fields. We do not find that space derivatives in the magnetic field lower the energy of the ground state as compared to a constant field configuration.

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I. INTRODUCTION

During the last few years flavor symmetry–breaking in (2+1)–dimensional QED has been at the center of interest for many authors. For example, Gusynin et al. [1] have demonstrated that uniform magnetic background fields act as a catalyst for dynamical flavor symmetry–breaking in 2+1 dimensions. Others [2] have looked at the impact of static inhomogeneous magnetic field configurations on the ground state of QED_{2+1}.

The problems attacked in the present work are quite similar; the methods employed, however, and the results obtained are somewhat different. In particular the results in the last chapter are in contradiction to the findings of previous works. We also want to work out some similarities and differences between QED_{2+1} and the real world of QED_{3+1}. This is done in section II. In section III we are mainly concerned with the QED_{2+1} vacuum structure and its (in-)stability with respect to non-constant magnetic fields. Our results are then summarized in section IV.

II. FERMI–CONDENSATE AND TRACE OF THE ENERGY–MOMENTUM TENSOR IN QED_{3+1} AND QED_{2+1}

Consider the Lagrangian of QED_{3+1}:

$$\mathcal{L} = -\bar{\psi}\left[m + \gamma\left(\tfrac{1}{i}\partial - eA\right)\right]\psi.\tag{1}$$

When we take the derivative of the generating functional,

$$Z = e^{iW} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, e^{-i\int d^4x\,\bar{\psi}\left[m+\gamma\left(\frac{1}{i}\partial-eA\right)\right]\psi},$$

i.e.,

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\begin{aligned}
\frac{\partial \ln Z}{\partial m} &= \frac{1}{Z}\frac{\partial Z}{\partial m}\\
&= \frac{-i}{Z}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A \left(\int d^4x\,\bar{\psi}(x)\psi(x)\right)e^{-i\int\bar{\psi}[\%]\psi}\\
&= -i\int d^4x\,\langle\bar{\psi}(x)\psi(x)\rangle
\end{aligned}
$$

or with $\ln Z = \mathrm{i} W : -\frac{\partial W}{\partial m} = \int \mathrm{d}^4 x \langle \bar\psi(x)\psi(x)\rangle$, we obtain

$$\langle \bar\psi\psi \rangle = -\frac{\partial \mathcal{L}^{(1)}}{\partial m} \,. \tag{2}$$

Here we introduced the one–loop effective Lagrangian $\mathcal{L}^{(1)}$ via

$$W^{(1)} = \int \mathrm{d}^4 x \, \mathcal{L}^{(1)}(x) \,.$$

When we employ the Green's function

$$G(x, x'|A) = \mathrm{i}\langle T(\psi(x)\bar\psi(x'))\rangle \,,$$

we can produce the following useful equalities:

$$\frac{\partial \mathcal{L}^{(1)}(x)}{\partial m} = -\mathrm{i}\,\,\mathrm{tr}\, G(x, x|A) = -\langle \bar\psi(x)\,\psi(x)\rangle \,. \tag{3}$$

A variant of the above procedure implies for the trace of the energy–momentum tensor

$$\langle \Theta_\mu{}^\mu(x) \rangle = -\mathrm{i}\, m\, \mathrm{tr}\, G(x, x|A) = m\frac{\partial \mathcal{L}^{(1)}(x)}{\partial m}$$
$$= -m\langle \bar\psi(x)\psi(x)\rangle \,. \tag{4}$$

With the aid of Schwinger's proper–time technique [3] we find for a constant magnetic field configuration, $B = B_3 = \mathrm{const.}$:

$$\langle \Theta_\mu{}^\mu \rangle(B) = \frac{eBm^2}{4\pi^2} \int\limits_0^\infty \frac{\mathrm{d}z}{z^2} \mathrm{e}^{-\frac{m^2}{eB}z} \left[z \coth z - 1 - \tfrac{1}{3}z^2 \right] \,. \tag{5}$$

After performing the integration, we get [4]

$$\langle \Theta_\mu{}^\mu \rangle(B) = -\frac{e^2 B^2}{12\pi^2} - \frac{m^4}{4\pi^2} \ln \frac{m^2}{2eB} + \frac{eBm^2}{4\pi^2} \ln \frac{m^2}{2eB}$$
$$+ \frac{m^4}{4\pi^2} + \frac{eBm^2}{2\pi^2} \left[\ln \Gamma\left(\tfrac{m^2}{2eB}\right) - \tfrac{1}{2} \ln 2\pi \right] \,.$$
$$\tag{6}$$

We can also identify

$$\langle \bar\psi\psi \rangle(B) = \frac{m^3}{4\pi^2} \ln \frac{m^2}{2eB} - \frac{eBm}{4\pi^2} \ln \frac{m^2}{2eB} - \frac{m^3}{4\pi^2}$$
$$- \frac{eBm}{2\pi^2} \left[\ln \Gamma\left(\tfrac{m^2}{2eB}\right) - \tfrac{1}{2} \ln 2\pi \right] \,. \tag{7}$$

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As a consistency check with Schwinger's formula for pair production [3], we obtain

$$\operatorname{Im}\langle\Theta_\mu{}^\mu\rangle(E) = m\frac{\partial}{\partial m}\operatorname{Im}\mathcal{L}^{(1)}(E) = -\frac{m^2 eE}{4\pi^2}\sum_{n=1}^\infty \frac{e^{-\frac{\pi m^2}{eE}n}}{n}. \tag{8}$$

Now we turn to a parity–invariant model of (2+1)–dimensional QED with two flavors ($\pm$). The dynamics is contained in the Lagrangian [5]

$$-\bar\psi[m + \gamma\left(\tfrac{1}{i}\partial - eA\right)]\psi = \mathcal{L} = \sum_{\lambda=\pm}\mathcal{L}_\lambda$$
$$= -\sum_{\lambda=\pm}\bar\psi_\lambda\left[\lambda m + \gamma\left(\tfrac{1}{i}\partial - eA\right)\right]\psi_\lambda. \tag{9}$$

On the left-hand side ψ is a four–component spinor and the γ's are 4×4 matrices whereas the $\psi_\pm$ are two–component spinors and the γ's on the right-hand side are 2×2 (Pauli–)matrices. We are working in 3–space with $x = (x^0, x^1, x^2)$ and $g = \operatorname{diag}(-1, 1, 1)$.

The fermion condensate can be written as

$$W(x; m) \equiv \langle\bar\psi(x)\psi(x)\rangle = \sum_{\lambda=\pm}\langle\bar\psi_\lambda(x)\psi_\lambda(x)\rangle_\lambda$$
$$= \sum_{\lambda=\pm}\lambda\,W_\lambda(x; m)\,,\ W_-(x; m) = W_+(x; -m)$$
$$= \sum_{\lambda=\pm}\lambda\,W_+(x; \lambda m) = W_+(x; m) - W_+(x; -m)\,, \tag{10}$$
$$W(x; -m) = -W(x; m)\,.$$

We are interested in the condensate described by the parity–invariant Lagrangian (9):

$$\langle\bar\psi(x)\psi(x)\rangle = \langle\psi^\dagger(x)\gamma^0_{4\times4}\psi(x)\rangle \quad,$$
$$\gamma^0_{4\times4} = \begin{pmatrix} \gamma^0_{2\times2} & 0 \\ 0 & -\gamma^0_{2\times2} \end{pmatrix}. \tag{11}$$

Changing the sign of $\gamma^0_{2\times2} = \sigma_3$ corresponds to changing the sign of m. For the moment we choose $m > 0$; but there will be a $\operatorname{sgn}(m)$ factor later on.

Earlier in this section we derived the relation

$$\langle\bar\psi(x)\psi(x)\rangle = i\operatorname{tr}G(x, x|A) = i\operatorname{tr}\int\frac{d^3k}{(2\pi)^3}\mathcal{G}(k)$$

$$= -\frac{1}{(2\pi)^3} \int d^3k \int_0^\infty ds\, e^{\left\{-is\left[m^2-k^{0^2}+\frac{\tan z}{z}k_\perp^2\right]\right\}}$$

$$\times \mathrm{tr}\left[\frac{e^{i\sigma_3 z}}{\cos z}\left(m+\gamma^0 k^0 - \frac{e^{-i\sigma_3 z}}{\cos z}(\gamma\cdot k)_\perp\right)\right]$$

$$z = eBs\,. \tag{12}$$

Here we employed formula (2.47) of ref. [6]. The momentum integral and the traces can readily be done with the intermediate result

$$\langle\bar\psi\psi\rangle = -\int_0^\infty \frac{ds}{8(\pi s)^{\frac{3}{2}}}4m(eBs)\cot(eBs)e^{-i\left(m^2 s+\frac{\pi}{4}\right)}\,. \tag{13}$$

To have a convergent expression on the right-hand side we need one subtraction which changes (13) into ($s = -it$):

$$\langle\bar\psi\psi\rangle(B) = -\frac{m}{2\pi^{\frac{3}{2}}}\int_0^\infty dt\, e^{-m^2 t}t^{-\frac{1}{2}}\left[eB\coth(eBt)-\frac{1}{t}\right]\,. \tag{14}$$

At this stage we need the value of the two integrals which after dimensional regularisation yield

$$I_1 = \int_0^\infty dt\, e^{-m^2 t}t^{-\frac{1}{2}}(eB)\coth(eBt)$$

$$= \sqrt{eB}\sqrt{\pi}\left[\sqrt{2}\,\zeta\left(\tfrac{1}{2},\tfrac{m^2}{2eB}\right)-\frac{\sqrt{eB}}{m}\right]\,,$$

$$I_2 = \int_0^\infty dt\, e^{-m^2 t}t^{-\frac{3}{2}} = 2\sqrt{\pi}m\,.$$

The use of these integrals produces

$$\langle\bar\psi\psi\rangle(B) = -\frac{1}{2\pi}\left[m\sqrt{2eB}\,\zeta\left(\tfrac{1}{2},1+\tfrac{m^2}{2eB}\right)+eB-2m^2\right]\,. \tag{15}$$

This result should be read side by side with formula (7).

We observe that for $m > 0$ and $(eB) > 0$ we obtain

$$\lim_{m\to 0_+}\langle\bar\psi\psi\rangle(B) = -\frac{eB}{2\pi} \tag{16}$$

and more generally

$$\lim_{m \to 0_+} \langle \bar{\psi}\psi \rangle(B) = -\text{sgn}(m)\frac{eB}{2\pi} \,. \tag{17}$$

Note that in 3+1 dimensions we would have obtained $\langle \bar{\psi}\psi \rangle \propto m \ln m = 0$ as $m \to 0$, i.e., there is no spontaneous flavor symmetry breaking in QED_{3+1}. Employing equation (4) we can make contact with the trace of the energy–momentum tensor in QED_{2+1}:

$$\langle \Theta_\mu{}^\mu \rangle = -m\langle \bar{\psi}\psi \rangle$$

$$= \frac{1}{2\pi}\left[m^2\sqrt{2eB}\,\zeta\left(\tfrac{1}{2}, 1 + \tfrac{m^2}{2eB}\right) + meB - 2m^3 \right] \,. \tag{18}$$

Unlike the result in 4–dimensional QED [4],

$$\lim_{m \to 0}\langle \Theta_\mu{}^\mu \rangle_{(4)} = -\frac{1}{12\pi^2}e^2 B^2 = -\frac{2\alpha}{3\pi}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

there is no trace-anomaly in 3–dimensional QED:

$$\lim_{m \to 0}\langle \Theta_\mu{}^\mu \rangle = 0 \,, \tag{19}$$

as it should.

Incidentally, the formula for the pair production rate in QED_{2+1}, which should be read together with expression (8), is given by

$$2\text{Im}\mathcal{L}^{(1)}(E) = \frac{1}{4\pi^2}(eE)^{\frac{3}{2}}\sum_{n=1}^{\infty}\frac{e^{-\frac{\pi m^2}{eE}n}}{n^{\frac{3}{2}}} \,. \tag{20}$$

III. QED_{2+1} FOR NON-CONSTANT FIELDS

We now turn to a disussion of external inhomogeneous electromagnetic fields interacting with massive fermions and begin by introducing the Hamiltonian $\not{D}^2$ which acts in a fictitious quantum mechanical Hilbert space and whose coordinate representation is given by

$$\langle x|H|y \rangle = (\not{D}_x^2)\delta(x - y) \,, \quad \not{D}_x = \gamma^\mu(\partial_x^\mu - \text{i}eA^\mu(x)) \,. \tag{21}$$

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We are working in Euclidean space and the γ–matrices are the above-mentioned 2×2 matrices $\gamma_{2\times2}$. The time developement in this 3–dimensional space is governed by the operator

$$K(t) = e^{-Ht}, \quad t > 0.$$

(22)

The well-known relation between the kernel $K(t)$ and the Green's function G is stated in

$$G = \int_0^\infty \mathrm{d}t\, e^{-m^2 t}\, K(t) = \int_0^\infty \mathrm{d}t\, e^{-(H+m^2)t}$$

$$= \frac{1}{\slashed{D}^2 + m^2}.$$

(23)

We will be interested in the matrix element

$$K(x, y; t) = \langle x, t | y, 0 \rangle = \langle x | e^{-Ht} | y \rangle$$

(24)

which satisfies the diffusion equation

$$\left(\frac{\partial}{\partial t} + \slashed{D}_x^2 \right) K(x, y; t) = 0$$

(25)

with $K(x, y; 0) = \langle x | y \rangle = \delta(x - y)$.

Introducing $\Pi_\mu = p_\mu - eA_\mu$ and $p_\mu = -i\partial_\mu$ we can rewrite the Hamiltonian in the form ($\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$)

$$H = \slashed{D}^2 = -(\gamma\Pi)^2 = \Pi_\mu \Pi_\mu - \tfrac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}$$

$$\text{or} \quad H = -\partial_\mu \partial_\mu + ie(\partial_\mu A_\mu) + ieA_\mu \partial_\mu$$

$$+ e^2 A_\mu A_\mu - \tfrac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}.$$

When this Hamiltonian is substituted into (25) we find for the diffusion equation in QED$_3$

$$\left(-\partial^2 + 2ieA_\mu \partial_\mu + X + \frac{\partial}{\partial t} \right) K(x, y; t) = 0,$$

(26)

$$\text{with} \quad X = ie\partial_\mu A_\mu + e^2 A_\mu A_\mu - \tfrac{e}{2}\sigma_{\mu\nu}F_{\mu\nu}.$$

For vanishing fields equation (26) can easily be solved by

$$K_0(x, y; t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{(x-y)^2}{4t}}.$$

(27)

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For non-vanishing fields we try to solve (26) by the ansatz

$$K(x, y; t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{(x-y)^2}{4t}} \sum_{k=0}^{\infty} a_k(x, y)\, t^k \,. \tag{28}$$

This expression, when inserted in (26), yields the following recursion relation for the coefficients a_k:

$$k = 0 : \quad (x-y)_\mu D_\mu a_0(x, y) = 0$$

$$k \geq 0 : \quad \left(D^2 + \tfrac{e}{2}\sigma F\right) a_k(x, y)$$

$$= (k+1)a_{k+1}(x, y)$$

$$+ (x-y)_\mu (\partial_\mu - \mathrm{i} e A_\mu) a_{k+1}(x, y) \,.$$

These recursion relations are formally the same as the ones that one encounters in QED$_4$ [7]. Furthermore, from our experience with constant field configurations we decompose $K(x, y; t)$ into a gauge independent factor and the counter-gauge factor

$$\Phi(x, y) = \exp\left\{ \mathrm{i} e \int_y^x \mathrm{d}\xi_\mu\, A_\mu(\xi) \right\} \,,$$

which allows us to separate off the gauge dependence from the coefficients a_k by writing

$$a_k(x, y) = \Phi(x, y)\, f_k(x, y) \,.$$

Now we take over the known results from QED$_4$ [7] and so obtain $([f_k](x) = \lim_{y \to x} f_k(x, y))$

$$[f_0](x) = 1 \quad , \quad [f_1](x) = \tfrac{e}{2}\sigma F \,,$$

$$[f_2](x) = \tfrac{e^2}{12} F^2 + \tfrac{e}{12}\sigma F^{,\mu\mu} + \tfrac{1}{2}\left[\tfrac{e}{2}\sigma F\right]^2 \,, \tag{29}$$

$$[f_3](x) = \tfrac{1}{3!}\left[\tfrac{e}{2}\sigma F\right]^3 + \tfrac{e}{2}\sigma F\left[\tfrac{e}{12}\sigma F^{,\mu\mu} + \tfrac{e^2}{12}F^2\right]$$

$$+ \tfrac{e^2}{48}\sigma F^{,\mu}\sigma F^{,\mu} + \tfrac{e}{120}\sigma F^{,\mu\mu\nu\nu} - \tfrac{e^2}{30}F^{\alpha\beta}F^{\alpha\beta,\mu\mu}$$

$$- \tfrac{e^2}{45}(F^{,\mu})^2 - \tfrac{e^2}{180}F^{\alpha\mu,\mu}F^{\alpha\nu,\nu} \,, \tag{30}$$

$$\text{where} \quad \sigma F = \sigma^{\alpha\beta}F^{\alpha\beta} \,, \ F^2 = F^{\alpha\beta}F^{\beta\alpha} \,,$$

$$(F^{,\mu})^2 = F^{\alpha\beta,\mu}F^{\alpha\beta,\mu} \text{ and } F^{\alpha\beta,\lambda} = \frac{\partial}{\partial x_\lambda}F^{\alpha\beta} \,.$$

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These $[f_k](x)$ contain contributions with and without space derivatives. Hence, it is only natural to write

$$\sum_{k=0}^{\infty}[f_k]\, t^k = (1 + b_1(x)\, t + b_2(x)\, t^2 + \ldots)\sum_{k=0}^{\infty}[f_k^c](x)\, t^k \,. \tag{31}$$

The coefficients $[f_k^c]$ depend on non-constant fields in the same manner as they would on constant fields. The $b_i(x)$ represent corrections for non-constant fields. According to (29) and (30) the terms without derivatives are given by

$$[f_1^c] = \tfrac{e}{2}\sigma F \quad , \quad [f_2^c] = \tfrac{e^2}{12}F^2 + \tfrac{1}{2}\left[\tfrac{e}{2}\sigma F\right]^2 ,$$
$$[f_3^c] = \tfrac{1}{3!}\left[\tfrac{e}{2}\sigma F\right]^3 + \tfrac{e}{2}\sigma F\,\tfrac{e^2}{12}F^2 \,. \tag{32}$$

Substituting the results of (32),(29) and (30) into (31) produces

$$b_1(x) = 0 \quad , \quad b_2(x) = \tfrac{e}{12}\sigma F^{,\mu\mu}$$
$$b_3(x) = \tfrac{e^2}{48}\sigma F^{,\mu}\sigma F^{,\mu} + \tfrac{e}{120}\sigma F^{,\mu\mu\nu\nu} \tag{33}$$
$$-\tfrac{e^2}{30}F^{\alpha\beta}F^{\alpha\beta,\mu\mu} - \tfrac{e^2}{45}(F^{,\mu})^2 - \tfrac{e^2}{180}F^{\alpha\mu,\mu}F^{\alpha\nu,\nu} \,.$$

The series in (31) containing the form of the constant field contributions can be summed up into the well-known result [3]

$$K^c(x,y;t) = \tfrac{1}{(4\pi t)^{\frac{3}{2}}}\Phi(x,y)\exp\left\{\tfrac{e}{2}\sigma Ft\right\} \tag{34}$$
$$\times \exp\left\{-\tfrac{1}{2}\,\mathrm{tr}\,\ln\left(\tfrac{\sin(eFt)}{eFt}\right)\right\}$$
$$\times \exp\left\{-\tfrac{1}{4}(x-y)eF\cot(eFt)(x-y)\right\} \,.$$

In the limiting case $x \to y$ we then obtain for non-constant fields:

$$K(x,x;t) = \tfrac{1}{(4\pi t)^{\frac{3}{2}}}\exp\left\{\tfrac{e}{2}\sigma Ft - \tfrac{1}{2}\,\mathrm{tr}\,\ln\left(\tfrac{\sin(eFt)}{eFt}\right)\right\}$$
$$\times \left[1 + \tfrac{e}{12}\sigma F^{,\mu\mu}t^2 + \left(\tfrac{e^2}{48}\sigma F^{,\mu}\sigma F^{,\mu}\right.\right. \tag{35}$$
$$\left.+ \tfrac{e}{120}\sigma F^{,\mu\mu\nu\nu} - \tfrac{e^2}{30}F^{\alpha\beta}F^{\alpha\beta,\mu\mu} - \tfrac{e^2}{45}(F^{,\mu})^2\right.$$
$$\left.\left.- \tfrac{e^2}{180}F^{\alpha\nu,\mu}F^{\alpha\nu,\nu}\right)t^3 + \ldots\right] \,.$$

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Finally we arrive at the (Euclidean) one–loop effective action for non-constant field configurations ($F_\mu^* = \frac{1}{2}\epsilon_{\mu\nu\rho}F_{\nu\rho}$):

$$W^{(1)} = \int \mathrm{d}^3x \, \mathcal{L}^{(1)}(x)$$

$$= \frac{1}{2} \int \mathrm{d}^3x \int\limits_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-m^2 t} \mathrm{Tr}[K(x,x;t) + \mathrm{c.t.}] \tag{36}$$

where

$$\mathrm{Tr}[K(x,x;t) + \mathrm{c.t.}] \tag{37}$$

$$= \frac{1}{(4\pi t)^{\frac{3}{2}}} \mathrm{tr}_\gamma \Big\{ \Big(eF^* t \coth(eF^* t) + \frac{e}{2}\sigma F t \Big)$$

$$\times \Big[1 + \frac{e}{12}\sigma F^{,\mu\mu} t^2 + \Big(\frac{e^2}{48}\sigma F^{,\mu}\sigma F^{,\mu} + \frac{e}{120}\sigma F^{,\mu\mu\nu\nu}$$

$$- \frac{e^2}{30} F^{\alpha\beta} F^{\alpha\beta,\mu\mu} - \frac{e^2}{45}(F^{,\mu})^2 - \frac{e^2}{180} F^{\alpha\mu,\mu} F^{\alpha\nu,\nu} \Big) t^3 \Big] \Big\} \,.$$

Deviating from QED$_4$ we now need the following trace identities valid for QED$_3$ only:

$$\mathrm{tr}_\gamma 1 = 2 \,, \ \mathrm{tr}\,\sigma^{\mu\nu} = 0 \,, \ \mathrm{tr}\,\sigma^{\mu\nu}\sigma^{\lambda\sigma} = 2(\delta^{\mu\lambda}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\lambda}) \,,$$

$$\mathrm{tr}\sigma^{\mu\nu}\sigma^{\lambda\sigma}\sigma^{\rho\kappa} = 2\mathrm{i}(\epsilon^{\mu\nu\rho}\epsilon^{\lambda\sigma\kappa} - \epsilon^{\mu\nu\kappa}\epsilon^{\lambda\sigma\rho}) \,.$$

Performing the various trace operations in (37) we finally end up with

$$W^{(1)} = \frac{1}{16\pi^{\frac{3}{2}}} \int\limits_0^\infty \frac{\mathrm{d}t}{t^{\frac{5}{2}}} \mathrm{e}^{-m^2 t} \int \mathrm{d}^3x \Big\{ 2(eF^* t \coth(eF^* t) - 1)$$

$$+ \frac{e^2}{15} eF^* t \coth eF^* t \Big(\frac{7}{12} F^{\alpha\beta,\mu} F^{\alpha\beta,\mu} - F^{\alpha\beta} F^{\alpha\beta,\mu\mu}$$

$$- \frac{1}{6} F^{\alpha\mu,\mu} F^{\alpha\nu,\nu} \Big) t^3 + \frac{e^2}{6} F^{\lambda\sigma} F^{\lambda\sigma,\mu\mu} t^3$$

$$+ \frac{e^2}{60} F^{\lambda\sigma} F^{\lambda\sigma,\mu\mu\nu\nu} t^4 \Big\} \,. \tag{38}$$

Note that all terms on the right-hand side yield finite contributions.

In the sequel we will mostly be interested in non-constant (static) external magnetic fields:

$$F^{12} = B \quad , \quad F^* = B > 0 \,. \tag{39}$$

Then the first term in the curly brackets of (38) yields the well-known form for the effective Lagrangian in QED$_3$ for constant B fields [8]:

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$$\mathcal{L}_{\text{eff}}^{(1)} = \frac{\sqrt{2}}{2\pi}(eB)^{\frac{3}{2}}\,\zeta\left(-\tfrac{1}{2},\tfrac{m^2}{2eB}\right) - \frac{meB}{4\pi}\,. \tag{40}$$

The correction terms for non-constant B fields that follow from (38) are contained in three contributions:

$$\mathcal{L}_{\text{A,eff}}^{(1)} = \frac{1}{16\pi^{\frac{3}{2}}}\frac{e^2}{15}\,Q\int\limits_0^\infty \mathrm{d}t\,\sqrt{t}(eBt)\coth(eBt)\,\mathrm{e}^{-m^2 t}\,, \tag{41}$$

where

$$Q = \left(\tfrac{7}{12}F^{\alpha\beta,\mu}F^{\alpha\beta,\mu} - F^{\alpha\beta}F^{\alpha\beta,\mu\mu} - \tfrac{1}{6}F^{\alpha\mu,\mu}F^{\alpha\nu,\nu}\right)$$
$$= \left(\tfrac{7}{6}\partial^\mu B\partial^\mu B - B\partial^2 B - \tfrac{1}{6}\partial^j B\partial^j B\right),\, j = 1, 2\,. \tag{42}$$

The value of the integral occurring in (41) turns out to be

$$\frac{1}{(eB)^{\frac{3}{2}}}\frac{3\sqrt{\pi}}{4}\left[\frac{1}{2\sqrt{2}}\zeta\left(\tfrac{5}{2},\tfrac{m^2}{2eB}\right) - \left(\frac{eB}{m^2}\right)^{\frac{5}{2}}\right]\,.$$

Hence we obtain

$$\mathcal{L}_{\text{A,eff}}^{(1)} = \frac{\sqrt{e}}{320\pi}\frac{Q}{B^{\frac{3}{2}}}\left[\frac{1}{2\sqrt{2}}\zeta\left(\tfrac{5}{2}, 1 + \tfrac{m^2}{2eB}\right) + \left(\frac{eB}{m^2}\right)^{\frac{5}{2}}\right]\,. \tag{43}$$

The last two terms in (38) contribute

$$\mathcal{L}_{\text{B,eff}}^{(1)} = \frac{1}{16\pi^{\frac{3}{2}}}\frac{e^2}{6}F^{\lambda\sigma}F^{\lambda\sigma,\mu\mu}\int\limits_0^\infty \mathrm{d}t\,\sqrt{t}\,\mathrm{e}^{-m^2 t}$$
$$= \frac{1}{96\pi}\frac{e^2}{|m|^3}(B\partial^2 B)\,, \tag{44}$$

$$\mathcal{L}_{\text{C,eff}}^{(1)} = \frac{1}{16\pi^{\frac{3}{2}}}\frac{e^2}{60}F^{\lambda\sigma}F^{\lambda\sigma,\mu\mu\nu\nu}\int\limits_0^\infty \mathrm{d}t\,t^{\frac{3}{2}}\mathrm{e}^{-m^2 t}$$
$$= \frac{1}{640\pi}\frac{e^2}{|m|^5}(B\partial^2\partial^2 B)\,. \tag{45}$$

Finally we come to the limiting case of vanishing fermion masses $(m \to 0)$. Let us regard the convergent terms of (40) and (43) first:

$$\mathcal{L}_{\text{eff}}^{(1)} = -\frac{1}{4\sqrt{2}\pi^2}\,(eB)^{\frac{3}{2}}\,\zeta\left(\tfrac{3}{2}\right)\,, \tag{46}$$

$$\mathcal{L}_{\text{A,eff}}^{(1)} = \frac{1}{4\sqrt{2}\pi^2}\frac{\pi}{160}\frac{\sqrt{e}\,Q}{B^{\frac{3}{2}}}\,\zeta\left(\tfrac{5}{2}\right)\,. \tag{47}$$

Now, for static inhomogeneous B fields we have

$$Q = \partial^j B \partial^j B - 2B\partial^2 B \tag{48}$$

so that

$$-\frac{1}{B^{\frac{3}{2}}}2B\partial^2 B = 2\partial_\mu\left(\frac{1}{B^{\frac{1}{2}}}\right)\partial^\mu B = -\frac{\partial^j B \partial^j B}{B^{\frac{3}{2}}} \tag{49}$$

which exactly cancels the first term in (48). Hence in the limit $m \to 0$ there are no corrections to the one–loop effective Lagrangian that arise from the inhomogeneity of the static B field.

There remains the problem with the diverging $(m \to 0)$ contributions contained in (43),(44) and (45). These can be made to vanish by formulating QED_{2+1} in a parity–invariant manner by adding another flavor degree of freedom as done in section 2. The result is a model with four-component spinors ψ, which is equivalent to a theory describing two species of two-component spinors $\psi_\pm$, one with mass $+m$ and the other with mass $-m$. So far we chose m to be positive. In our new extended version of QED_{2+1} we now encounter the mass term

$$-m\bar\psi\psi = -m\psi^\dagger\gamma^0_{4\times4}\psi = -m\psi^\dagger_+\gamma^0_{2\times2}\psi_+ + m\psi^\dagger_-\gamma^0_{2\times2}\psi_-$$

$$\text{where we used } \gamma^\mu_{4\times4} = \begin{pmatrix} \gamma^\mu_{2\times2} & 0 \\ 0 & -\gamma^\mu_{2\times2} \end{pmatrix} \text{ and } \gamma^0_{2\times2} = \sigma_3.$$

The terms in the effective Lagrangian that remain finite (after taking the limit $m \to 0$) carry the same sign. The others, however, that vanish or are singular for $m \to 0$ have their origin in the mass term of the original Lagrangian and thus contain different signs for the two different two-component fermion fields. Since they have the same modulus they cancel each other.

IV. CONCLUSION

In the first half of this paper we studied the Fermi condensate in QED_{2+1}. We showed that the flavor condensate is non-zero as the fermion mass goes to zero. This is a specific

(2+1)–dimensional phenomenon since there is no spontaneous symmetry breaking in ordinary QED_{3+1}. We also compared the trace anomaly of the energy–momentum tensor and the electron–positron pair production rate in both theories.

In the second half we employed the heat kernel expansion to determine the Seeley coefficients that are necessary to calculate the effective action of QED_{2+1}. We investigated the QED_{2+1} ground state by allowing the static external magnetic field to become space dependent. Contrary to the result of reference [2] we find that the ground state probed in one-loop approximation is not shifted towards a lower value by the presence of an inhomogeneous static magnetic field – for $m \to 0$. Our calculations do not exhibit an instability of the uniform magnetic field state towards a more disordered state with inhomogeneous magnetic fields. Although we believe that the stable QED_{2+1} ground state tested with an external constant magnetic field is only an intermediate step on the way to the true ground state, we do not find – to one–loop order – that space inhomogeneities in the magnetic field change the situation.

REFERENCES

[1] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Phys.Rev.Lett. **73**, (1994) 3499; Phys.Rev. **D 52**, (1995) 4718

[2] D. Cangemi, E. D'Hoker and G. Dunne, Phys.Rev. **D 51**, (1995) R2513

[3] J. Schwinger, Phys.Rev. **82**, (1951) 664

[4] W. Dittrich and M. Sieber, J.Phys. **A 21**, (1988) L711

[5] G. Dunne and T. Hall, Phys.Rev., **D 53**, (1996) 2220 ;
R.R. Parwani, Phys.Lett. **358 B**, (1995) 101

[6] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Lecture Notes in Physics **220**, Springer–Verlag (1985)

[7] J. Hauknes, Ann.Phys. (N.Y.) **156**, (1984) 303

[8] S.K. Blau, M. Visser and A. Wipf, Int.J.Mod.Phys. **A 6**, (1991) 5409